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Continuity of separately continuous group actions in p-spaces

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Abstract

Let $f:X \times Y \to Z$ be a separately continuous mapping, where X is a Baire p-space and Z a completely regular space, and let $y \in Y$ be a q-point. We show that (i) f is strongly quasicontinuous at each point of $X \times \{y\}$, and (ii) if Z is a p-space, then f is subcontinuous at each point of $A \times \{y\}$, where A is a dense subset of X. Then, we use (i) and (ii) to prove that every separately continuous action of a left topological group, which is a Baire p-space, in a p-space, is a continuous action. In particular, every semitopological group, which is a Baire p-space, has a continuous multiplication.

Keywords: p-space; q-space; Separate continuity; Group action; Strong quasicontinuity; Subcontinuity; Semitopological group

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1. Introduction

In 1957, Ellis showed in [5] the following (Ellis theorem): Let G be a semitopological locally compact group and X a locally compact space; then every separately continuous action of G in X is continuous. Using this result, he then showed that every semitopological locally compact group is a topological group. Later in [14] Namioka showed that it suffices, in Ellis theorem, to suppose that G is right topological and countably Čech-complete regular, the space X being locally compact. In [2, Corollary 3.5] we have showed that Ellis theorem can be extented as follows: Every separately continuous action of a left topological group G, which is a Baire p-space, in a paracompact p-space X is continuous. This result enables us to give in [2] a partial positive answer to a problem of Pfister, who asked in [15, Remarks] whether semitopological Čech-complete groups have continuous multiplication or not. (This problem was settled after the works of Brand

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[4], Ellis [5], Montgomery [13] and Zelazko [17].) But, because of the paracompactness assumption on X, [2, Corollary 3.5] was not strong enough to solve this problem in the whole.

More recently, we gave in [3] by a direct method a complete positive answer to Pfister's problem. The purpose of this note is to show that it is also possible to extend Ellis theorem to the case where the component X is a p-space not necessarily paracompact. More precisely, we use the notion of strong quasicontinuity (introduced in [11], see also [10]) to prove the following: Every separately continuous action of a left topological group G, which is a Baire p-space, in a p-space X, is a continuous action (Theorem 4). It follows from this result that every semitopological group G, which is a Baire p-space, is a paratopological group, that is, the multiplication of G is continuous. Since Čech-complete spaces are p-space, this last result gives another way to solve Pfister's problem.

Theorem 4 is obtained as a particular case of a slightly more general statement (Theorem 3), where the left topological group is supposed to be only point-wise countably complete Baire space. The class of point-wise countably complete spaces is introduced in Section 2. This class contains every countably Čech-complete regular space and the following generalized metric spaces: p-spaces, $w\Delta$-spaces and M-espaces.

This paper is organised as follows. Section 3 is devoted to some auxiliary results on separately continuous mapping. Theorem 3 is then proved in Section 4, where some other consequences are also settled. In Section 2, we collect some notations and terminology.

2. Preliminaries

Let X, Y and Z be topological spaces.

(1) The space X is called a p-space [9] if X is completely regular and if there is a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of X such that the following condition is satisfied:

For each $n \in \mathbb{N}$ let $U_n \in \mathcal{U}_n$; if $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$, then $K = \bigcap_{n\in\mathbb{N}} \overline{U_n}$ is compact and the sequence $(\bigcap_{i\leqslant n} \overline{U_i})_{n\in\mathbb{N}}$ is an outer network for K (i.e., if U is an open subset of X containing K then there is $n \in \mathbb{N}$ such that $\bigcap_{i\leqslant n} \overline{U_i} \subset U$).

The class of p-spaces was originally introduced by Arhangel'skiĭ [1] in a different but equivalent form.

(2) A point x of X is a q-point [12] if there is a sequence of neighbourhoods $(O_n)_{n\in\mathbb{N}}$ of x in X such that, if $x_n \in O_n$ for each $n \in \mathbb{N}$, then the sequence $(x_n)_{n\in\mathbb{N}}$ has a cluster point in X. If all elements of X are q-points, one says that X is a q-space. Every p-space is a q-space [12].

(3) We shall say that the space X is point-wise countably complete if there is a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of X such that the following condition is satisfied:

Let $(A_n)_{n\in\mathbb{N}}$ a decreasing sequence of nonempty subsets of X; suppose that for each $n \in \mathbb{N}$, A_n is contained in at last an element U_n of $\mathcal{U}_n$ and that $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$; then $\bigcap_{n\in\mathbb{N}} \overline{A_n} \neq \emptyset$.

It follows immediately from the definitions (see [9]) that $w\Delta$-spaces, M-spaces and p-spaces (hence all Čech-complete spaces and all metric spaces) are point-wise countably complete. This is also the case of the strongly countably complete regular spaces [6], considered by Namioka in [14] and renamed countably Čech-complete regular spaces in [10].

(4) A mapping $f:X \to Y$ is subcontinuous at $x \in X$ if, for each net $(x_\alpha)_{\alpha\in A}$ in X which converges to x, the net $(f(x_\alpha))_{\alpha\in A}$ has a cluster point in Y. The mapping $f:X \to Y$ is subcontinuous [8] if it is subcontinuous at all points of X. Following [11] (see also [10]), we say that a mapping $f:X \times Y \to Z$ is strongly quasicontinuous at $(x,y) \in X \times Y$ if, for each neighbourhood W of $f(x,y)$ in Z and for each product of open sets $U \times V \subset X \times Y$ containing (x,y), there is a nonvoid open set $O \subset U$ and a neighbourhood V_1 of y in Y such that $f(O \times V_1) \subset W$.

3. Strong quasicontinuity and subcontinuity of a separately continuous mapping

Recall that a mapping $f:X \times Y \to Z$, where X, Y and Z are topological spaces, is separately continuous if, for each $x \in X$ (respectively for each $y \in Y$) the mapping $y \in Y \to f(x,y) \in Z$ (respectively the mapping $x \in X \to f(x,y) \in Z$) is continuous. In this section we establish two results related to separately continuous mappings. The theorems given below provide the framework for proving the main result of this paper (Theorem 3).

Theorem 1. *Let X be a point-wise countably complete Baire space, Y a topological space and Z a completely regular space. Let $f:X\times Y \to Z$ be a separately continuous mapping. Then, for each q-point b in Y, the mapping f is strongly quasicontinuous at each point of $X \times \{b\}$.*

Proof. Let $a \in X$, W an open subset of Z containing $f(a,b)$ and $U \times V \subset X \times Y$ a product of open sets containing (a,b). Let us show that there is a nonvoid open set $U' \subset U$ and a neighbourhood V' of b in Y such that $f(U' \times V') \subset W$. Suppose that this is impossible. Let $\phi: Z \to \mathbb{R}$ be a continuous function such that $\phi(f(a,b)) = 1$ and $\phi(W^c) \subset \{0\}$. (Recall that Z is completely regular.) Let ψ denotes the separately continuous function $\phi \circ f: X \times Y \to \mathbb{R}$. Fix a sequence $(O_n)_{n\in\mathbb{N}}$ of neighbourhoods of b in Y such that, if $y_n \in O_n$ for each $n \in \mathbb{N}$, then the sequence $(y_n)_{n\in\mathbb{N}}$ has a cluster point in Y. Finally, let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of open covers associated to the point-wise countably complete space X.

We are going to define a strategy σ for player β in Choquet game on the space X (see [16] for the definition of this game). Put $\sigma(\emptyset) = U \cap \{x \in X \mid \psi(x,b) > 1/2\}$ and let $x_0 \in X$. At the nth stroke, if α has chosen $V_1, \ldots, V_n$, then player β picks a point

$$y_n \in O_n \cap \left(\bigcap_{i<n} \{y \in Y \mid |\psi(x_i,y) - \psi(x_i,b)| < 1/n\}\right), \quad x_n \in V_n,\ G_n \in \mathcal{U}_n$$

and an open subset U_n of X satisfying $x_n \in U_n \subset \overline{U_n} \subset G_n \cap V_n$, such that $\psi(x_n, y_n) = 0$; and then β plays the following nonvoid open subet of V_n:

$$\sigma(V_1, \ldots, V_n) = \{x \in U_n \mid |\psi(x, y_n)| < 1/3\}.$$

Since X is a Baire space, σ cannot be a winning strategy (see [16]). Hence, there exists a winning game, say $(V_n)_{n\in\mathbb{N}}$, for player α against σ. As $\emptyset \neq \bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} U_n \subset \bigcap_{n\in\mathbb{N}} G_n$ and since, for each $n \in \mathbb{N}$, we have $\{x_i \mid i \geqslant n\} \subset G_n$, the sequence $(x_n)_{n\in\mathbb{N}}$ has a cluster point $x \in X$. Since we also have $\overline{U_{n+1}} \subset U_n$ for each $n \in \mathbb{N}$, the point x belongs to $\bigcap_{n\in\mathbb{N}} U_n$. Let $y \in Y$ be a a cluster point of the sequence $(y_n)_{n\in\mathbb{N}}$. We obtain, as it is easy to verify, that $\psi(x, y) \leqslant 1/3$ and $\psi(x, y) = \psi(x, b) \geqslant 1/2$, which is absurd. Hence, f is strongly quasicontinuous at (a, b). □

Theorem 2. *Let X be a point-wise countably complete Baire space, Y a topological space, Z a p-space and $f : X \times Y \to Z$ a separately continuous mapping. Then, for each q-point $b \in Y$, the set of points $x \in X$ such that f is subcontinuous at (x, b) is a dense subset of X.*

Proof. Let b be a q-point of Y. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a sequence of open covers associated to the p-space Z. For each $n \in \mathbb{N}$ let A_n be the set of all points $x \in X$ for which there is a neighbourhood V of (x, b) in $X \times Y$ and $U \in \mathcal{U}_n$ such that $f(V) \subset U$. Let us first show that the open set A_n is dense in X. Let O be a nonvoid open subset of X and let $x \in O$. Let $U \in \mathcal{U}_n$ such that $f(x, b) \in U$; there is, by Theorem 1, a nonvoid open set $O_1 \subset O$ and a neighbourhood O_2 of b in Y such that $f(O_1 \times O_2) \subset U$. It is clear that $O_1 \subset A_n$. Now, let $A = \bigcap_{n\in\mathbb{N}} A_n$. Since X is a Baire space, A is a dense G_δ subset of X; to conclude the proof it suffices to show that f is subcontinuous at each element of $A \times \{b\}$. Take $a \in A$ and let $((x_\alpha, y_\alpha))_{\alpha\in\Lambda}$ be a net in $X \times Y$ that converges to (a, b). Put $\mathcal{F} = \{\{f(x_\alpha, y_\alpha) \mid \beta \leqslant \alpha\} \mid \beta \in \Lambda\}$; we must show that $\bigcap\{\overline{F} \mid F \in \mathcal{F}\} \neq \emptyset$. For each $n \in \mathbb{N}$ let $U_n \in \mathcal{U}_n$ such that $f(a, b) \in U_n$, and $\beta_n \in \Lambda$ such that $\{f(x_\alpha, y_\alpha) \mid \beta_n \leqslant \alpha\} \subset U_n$. Suppose that $\bigcap\{\overline{F} \mid F \in \mathcal{F}\} = \emptyset$. Since the subspace $K = \bigcap_{n\in\mathbb{N}} \overline{U_n}$ of Z is a compact having the sequence $(\bigcap_{i\leqslant n} \overline{U_i})_{n\in\mathbb{N}}$ as an outer network, there is $F_0, \ldots, F_n \in \mathcal{F}$ and $k \in \mathbb{N}$ such that $K \subset \bigcap_{i\leqslant k} \overline{U_i} \subset \bigcup_{i\leqslant n} \overline{F_i}^{\,c}$; in particular, we have

$$\left(\bigcap_{i\leqslant k} \{f(x_\alpha, y_\alpha) \mid \beta_i \leqslant \alpha\}\right) \cap \left(\bigcap_{i\leqslant n} F_i\right) = \emptyset,$$

which is impossible since $\mathcal{F}$ is a filter-base. Hence $\bigcap\{\overline{F} \mid F \in \mathcal{F}\} \neq \emptyset$ as claimed. □

4. Continuity of a group action in a p-space

Let G be a group equipped with a topology. The group G is called left topological if, for each $g \in G$, the left translation $h \in G \to gh \in G$ is continuous. If all left translations and all right translations of G are continuous, G is called semitopological;

and, if the product mapping $(g,h) \in G \times G \to gh \in G$ is continuous, G is called paratopological. An action of G in a set X is a mapping $(g,x) \in G \times X \to gx \in X$ such that $g(hx) = (gh)x$, $ex = x$, for each $g, h \in G$ and for each $x \in X$, where e is the neutral element of G.

Theorem 3. *Let G be left topological group and suppose that G is point-wise countably complete Baire space. Then every separately continuous action of G in a p-space X, is a continuous action.*

Proof. Let $\varphi : G \times X \to X$ be a separately continuous action of G in a p-space X. Let $((g_\alpha, x_\alpha))_{\alpha \in \Lambda}$ be a net in $G \times X$ which converges to $(g,x) \in G \times G$, and let us show that gx is a cluster point of the net $(g_\alpha x_\alpha)_{\alpha \in \Lambda}$. This would imply that φ is continuous. Consider, by Theorem 2, a point $a \in G$ such that φ is subcontinuous at (a,x). Since the group G is left topological, we have $\lim ag^{-1}g_\alpha = a$, hence the net $(ag^{-1}g_\alpha x_\alpha)_{\alpha \in \Lambda}$ has a cluster point in X. It follows from the separate continuity of φ that the net $(g_\alpha x_\alpha)_{\alpha \in \Lambda}$ has a cluster point $y \in X$. To conclude the proof we show that $gx = y$; let V be a closed neighbourhood of gx in X; it suffices to show that $y \in V$. Denote by A the union of all open subsets ω of G for which there is a neighbourhood V_ω of x such that $\varphi(\omega \times V_\omega) \subset V$. It results from the strong quasicontinuity of φ at (g,x) (Theorem 1) that $g \in \overline{A}$. Let $a \in A$. We have $\lim ag^{-1}g_\alpha = a$, hence $ag^{-1}g_\alpha x_\alpha \in V$ for α big enough. Consequently, we have $ag^{-1}y \in V$ for each $a \in A$; since $g \in \overline{A}$, we obtain that $y \in V$. □

Since each p-space is point-wise countably complete, the following result is a particular case of Theorem 3.

Theorem 4. *Let G be a left topological group. If G is a Baire p-space, then each separately continuous action of G in a p-space X is a continuous action.*

Corollary 5. *Every semitopological Baire p-space is a paratopological group.*

Proof. Let G be such a group. The product mapping $(g,h) \in G \times G \to gh \in G$ is a separately continuous action of G in G, hence Theorem 4 applies. □

Corollary 6. *Every Čech-complete semitopological group G is a topological group.*

Proof. Since every Čech-complete space is a p-space, Corollary 5 implies that G is paratopological; hence, by Brand's result [4], the inversion of G is continuous. □

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